Super-resolution Radar

Reinhard Heckel

IBM Research (before: ETH Zurich)

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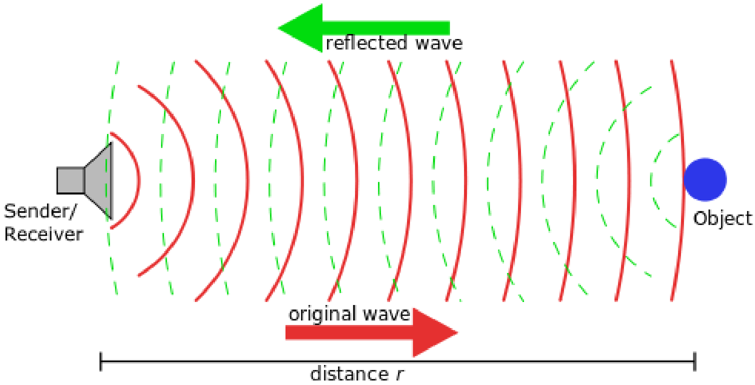
Joint work with:
V. Morgenshtern and M. Soltanolkotabi

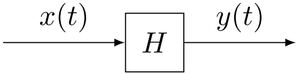
Motivation: Radar



Goal: determine location and velocities of objects

Model

Time-varying linear system H:



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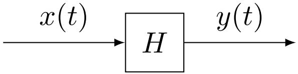

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Suppose the spreading function consists of S point scatterers:

$$y(t) = \sum_{j=1}^{S} b_j x(t - \bar{\tau}_j) e^{i2\pi\bar{\nu}_j t}$$

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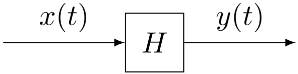

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Goal: Determine the triplets $(b_j, \bar{\tau}_j, \bar{\nu}_j)$ from I/O-measurement

Band and time-limitation

In practice: $x(t)$ bandlimited to $[0, B]$ and $y(t)$ observed on $[0, T]$

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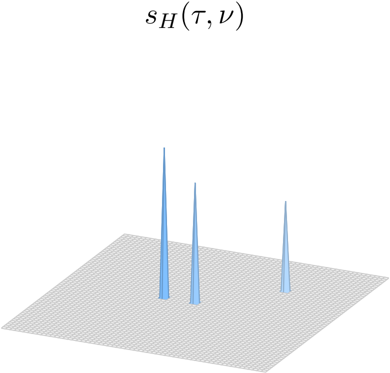

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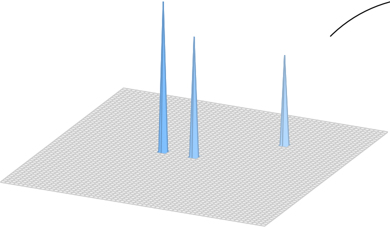
$s_H(\tau, \nu)$

$\operatorname{sinc}(\tau B)\operatorname{sinc}(\nu T) * s_H(\tau, \nu)$



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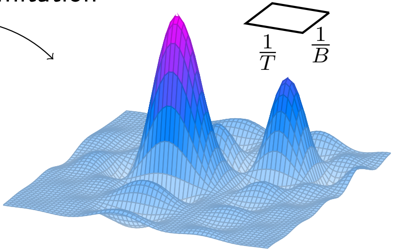

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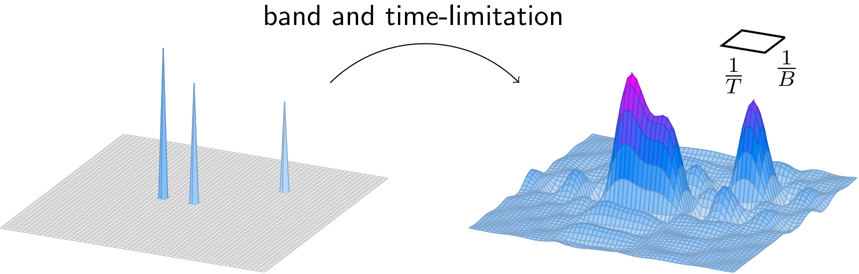

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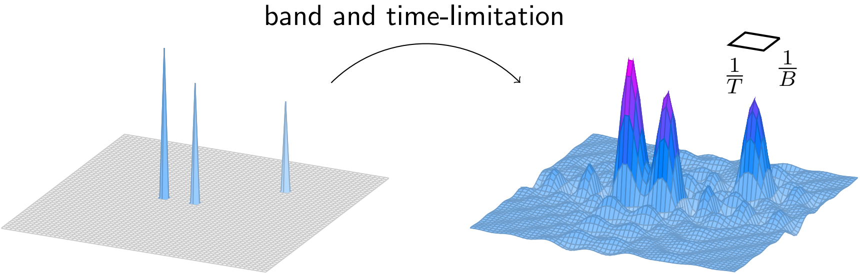

Standard pulse-Doppler radar

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- Transmits a train of band-limited pulses
- Samples $y(t)$ are rate $1/B$
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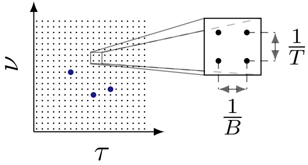
Delay-Doppler pairs are resolved only on a grid:



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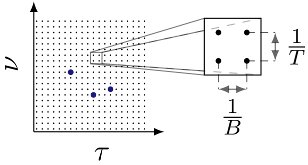

Resolution achieved is $\left(\frac{1}{B}, \frac{1}{T}\right)$!

The aim of this talk

Show that we can recover the delay-Doppler shifts **perfectly** via convex optimization even when

- x is bandlimited to $[0, B]$ and y is only observed on $[0, T]$
- the delays and Doppler shifts are large, i.e.,

$$(\bar{\tau}_j, \bar{\nu}_j) \in [0, T] \times [0, B]$$

Discretization through band and time limitation

Let $L = BT$

$$x(t) = \sum_{\ell=0}^{3L-1} x_\ell \operatorname{sinc}(tB - \ell), \quad x_\ell \text{ is } L \text{ periodic}$$

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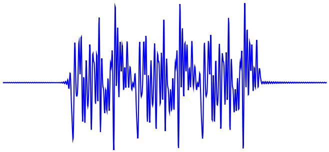
$x(t)$



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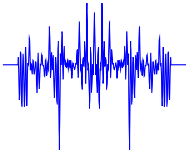

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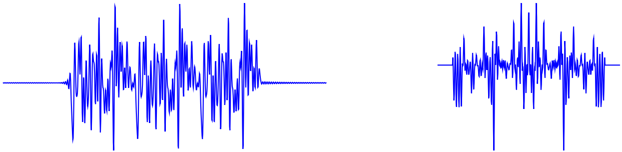

Due to path loss and finite velocity of the targets we may assume:

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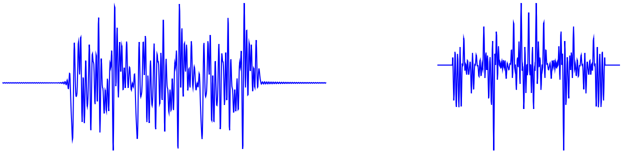

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$$(\bar{\tau}_j, \bar{\nu}_j) \in [0, T] \times [0, B]$$

$y(t)$ is band and essentially time limited as well, and on the order of BT-dimensional

Discretization through band and time limitation

Sampling at rate $1/B$ in the interval $[0, T]$ yields:

$$y_p = \sum_{j=1}^{S} b_j [\mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}]_p, \quad p = 0, ..., L-1$$

where

$$\tau_j = \frac{\bar{\tau}_j}{T} \in [0, 1], \quad \nu_j = \frac{\bar{\nu}_j}{B} \in [0, 1], \quad \mathbf{x} = [x_0, ..., x_{L-1}]$$

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$$\mathcal{T}_\tau \mathbf{x} = \mathbf{F}^H \begin{bmatrix} \ddots & & \\ & e^{-i2\pi k\tau} & \\ & & \ddots \end{bmatrix} \mathbf{F}\mathbf{x}$$

$\mathbf{F}$: discrete Fourier transform

The super-resolution Radar problem

Determine the **continuous** time-frequency shifts $(\tau_j, \nu_j) \in [0,1]^2$ from the samples

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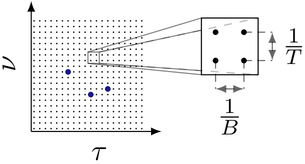
This talk: Provably recovery of the time-frequency pairs via convex optimization

Compressed sensing Radar [*Herman & Stromer, 2009*]

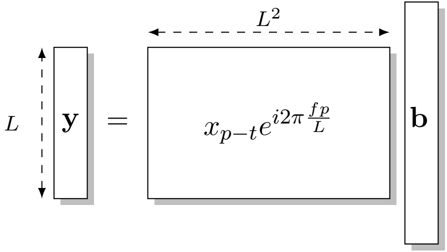
Suppose the $(\bar{\tau}_j, \bar{\nu}_j)$ lie on a $\left(\frac{1}{B}, \frac{1}{T}\right)$-grid



Then $\tau_j L$ and $\nu_j L$ are integers:

$$y_p = \sum_{j=1}^{S} b_j x_{p-(\tau_j L)} e^{i2\pi \frac{(\nu_j L)p}{L}} \quad p = 0, ..., L-1$$

Compressive sensing Radar



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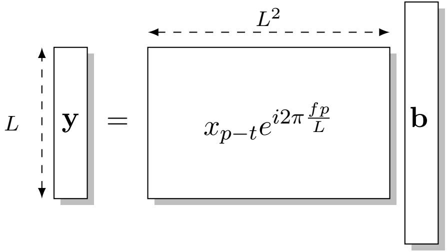

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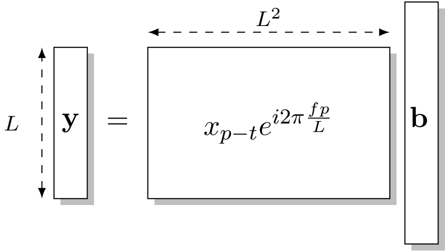

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- Recovery via ℓ_1 minimization provably succeeds provided that [*Krahmer et al. 2014*]

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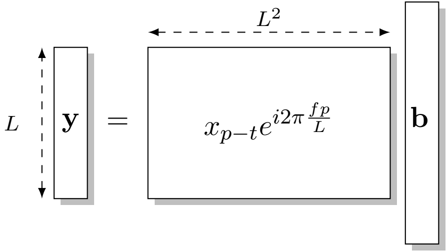

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- Resolution achieved by compressive sensing Radar is $\left(\frac{1}{B}, \frac{1}{T}\right)$!

Special case: Frequency shifts only

If $\tau_j = 0$, the problem reduces to a line spectral estimation problem:

$$y_p = x_p \sum_{j=1}^{S} b_j e^{i2\pi p \nu_j}, \quad p = 0, ..., L-1$$

Special case: Time shifts only

If $\nu_j = 0$, the problem reduces to a line spectral estimation problem in the Fourier domain:

$$Y_p = X_p \sum_{j=1}^{S} b_j e^{i2\pi p \tau_j}, \quad p = 0, ..., L-1$$

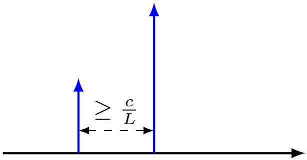
Recovery approaches for line spectral estimation

- **Classical:** Proney's method, MUSIC, ESPRIT

Recovery approaches for line spectral estimation

- **Classical:** Proney's method, MUSIC, ESPRIT
- **Modern:** Recovery via convex optimization [*Candès & Fernandez-Granda, 2014*]: Recovery is possible provided the minimum separation condition holds:

$$|\nu_j - \nu_i| \geq \frac{4}{L}, \quad \text{for all } j \neq i$$



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Some sort of separation between the time-frequency shifts is necessary!

Recovery via convex programming

Recovery via "ℓ_1-minimization":

$$\text{minimize} \sum_j |b_j| \text{ subject to } \mathbf{y} = \sum_j b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}$$

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- If the (τ_j, ν_j) lie on an **arbitrarily fine grid**, this is standard ℓ_1-minimization
- Formal versions for continuous (τ_j, ν_j):
  - Total-variation norm minimization [*Candès & Fernandez-Granda, 2014*]
  - Atomic norm minimization [*Bhaskar et al., 2013*], [*Tang et al., 2013*]

Recovery results for the super-resolution Radar problem

Main result

- Random probing signal: x_ℓ i.i.d. $\mathcal{N}(0, 1/L)$
- Random signs: sign of b_j is i.i.d. uniform on the real or complex unit sphere

Theorem

Let $\mathbf{y} = \sum_{j=1}^{S} b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}$. *With probability* $\geq 1 - \delta$, $(\tau_j, \nu_j, b_j), j = 1, ..., S$, *is the unique solution to* ℓ_1*-minimization if*

$$|\tau_j - \tau_i| \geq \frac{5}{L} \text{ or } |\nu_j - \nu_i| \geq \frac{5}{L}, \text{ for all } i \neq j$$

and if

$$S \leq cL \log^{-3}\left(\frac{L^6}{\delta}\right)$$

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- Essentially optimal

Implications for the continuous setup

The delay-Doppler shifts $\bar{\tau}_j, \bar{\nu}_j$ can be recovered perfectly provided that:

$$|\bar{\tau}_j - \bar{\tau}_i| \geq \frac{5}{B} \text{ or } |\bar{\nu}_j - \bar{\nu}_i| \geq \frac{5}{T}, \text{ for all } i \neq j$$

and

$$S \leq cBT \log^{-3}\left(\frac{(BT)^6}{\delta}\right)$$

Optimality condition

Lemma

Let $\mathbf{y} = \sum_{j=1}^{S} b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}$. *If there exists a polynomial of the form*

$$Q(\tau, \nu) = (\mathcal{F}_{\nu} \mathcal{T}_{\tau} \mathbf{x})^H \mathbf{q}$$

such that Q *is* **bounded** *and* **interpolates** *the sign pattern of the signal:*

$$Q(\tau_j, \nu_j) = \mathrm{sign}(b_j), \quad \textit{for all } j = 1, ..., S$$

$$|Q(\tau, \nu)| < 1, \quad \textit{for all other } (\tau, \nu)$$

then (b_j, τ_j, ν_j) *is the unique minimizer to* ℓ_1*-minimization.*

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Q certifies the optimality of (b_j, τ_j, ν_j)

Proof of optimality condition

For simplicity, suppose the τ_j, ν_j lie on a grid

Let $\mathbf{y} = \sum_{j=1}^{S} b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}$

For all a_k, τ_k, ν_k with $\mathbf{y} = \sum_k a_k \mathcal{F}_{\nu_k} \mathcal{T}_{\tau_k} \mathbf{x}$:

$$\sum_k |a_k| \geq \sum_k |b_k| + \mathrm{Re} \sum_k Q^*(\tau_k, \nu_k)(a_k - b_k)$$

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\end{aligned}$$

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&= \sum_k |b_k|
\end{aligned}$$

Construction of certificate Q

Proof is inspired by the construction of related certificates:

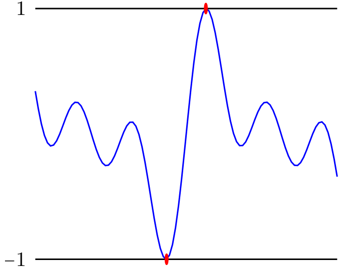
- Candès and Fernandez-Granda, "Towards a Mathematical Theory of Super-Resolution"
- Tang et al., "Compressed Sensing Off the Grid"

Certificate for $2D$ line spectral estimation

[*Candès and Fernandez-Granda, 2014*]:
For all b_j and (τ_j, ν_j) satisfying the **minimum separation condition** there exists a polynomial:

$$\bar{Q}(\tau, \nu) = \sum_{k,p=-L/2}^{L/2} \bar{q}_{k,p} e^{-i2\pi(k\tau + p\nu)}$$

that is **bounded** and **interpolates** $\mathrm{sign}(b_j)$ at (τ_j, ν_j)



Construction of certificate for $2D$ line spectral estimation

[*Candès and Fernandez-Granda, 2014*]:
Construct $\bar{Q}$ explicitly by interpolating $\operatorname{sign}(b_j)$ with a kernel $\bar{G}$ (and its partial derivatives $\bar{G}^{(1,0)}, \bar{G}^{(0,1)}$):

$$\bar{Q}(\tau,\nu) = \sum_{j=1}^{S} \alpha_j \bar{G}(\tau - \tau_j, \nu - \nu_j) + ...$$

Interpolation with fast-decaying low-frequency kernel $\bar{G}(\tau,\nu)$:

Construction of certificate for time-frequency shifts

Interpolation with functions $G_{(\tau_j,\nu_j)}(\tau,\nu)$ that reach their maxima close to (τ_j,ν_j) and decay fast around its maxima

Interpolating functions

Q is a $2D$ trigonometric polynomial with certain coefficients:

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$$Q(\tau,\nu) = (\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^H \mathbf{q} = \sum_{k,\ell=-L/2}^{L/2} [\mathbf{F}\mathbf{G}^H \mathbf{q}]_{p,k} e^{-i2\pi(k\tau+p\nu)}$$

- $\mathbf{G} \in \mathbb{C}^{L\times L^2}$ is the Gabor matrix with columns $\mathcal{F}_{p/L}\mathcal{T}_{k/L}\mathbf{x}$
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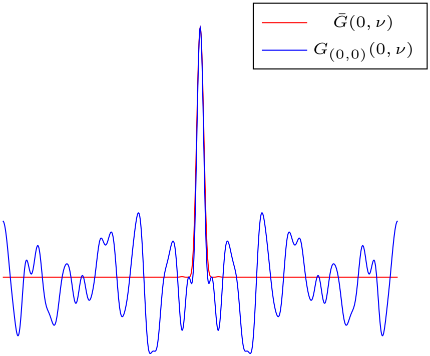
$$G_{(\tau_j,\nu_j)}(\tau,\nu) = \underbrace{[\ldots e^{i2\pi(\tau r + \nu q)} \ldots]\mathbf{F}\mathbf{G}^H}_{(\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^H} \mathbf{G}\mathbf{F}^H \begin{bmatrix} \vdots \\ g_r g_q e^{-i2\pi(\tau_j r + \nu_j q)} \\ \vdots \end{bmatrix}$$

g_k are coefficients of fast decaying low-frequency kernel

Interpolating functions

$$\mathbb{E}\left[G_{(\tau_j,\nu_j)}(\tau,\nu)\right] = \bar{G}(\tau-\tau_j,\nu-\nu_j)$$



Final steps of the proof

Step 1: With high probability there exist interpolation coefficients such that G (and its derivatives) interpolate $\mathrm{sign}(b_j)$

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Step 2: With those interpolations coefficients, Q is bounded with high probability uniformly for all (τ, ν) not in the support set

Practical implementations

Recovery of the time-frequency shifts via the dual

The coefficient vector $\mathbf{q}$ is a solution to the dual of ℓ_1-minimization:

$$\text{maximize}_{\mathbf{q}} \mathrm{Re}\langle \mathbf{q}, \mathbf{y}\rangle \text{ subject to } \sup_{\tau,\nu} |(\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^H \mathbf{q}| \leq 1$$

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- Dimensions of the matrices in this characterization are unspecified, fixing them leads to a relaxation

Recovery of the time-frequency shifts via the dual

Algorithm:

1. Solve the semidefinite programming relaxation to obtain $\mathbf{q}$
2. Identify the time frequency shifts as:
   $\{(\tau, \nu)\colon |(\mathcal{F}_\nu \mathcal{T}_\tau \mathbf{x})^H \mathbf{q}| = 1\}$

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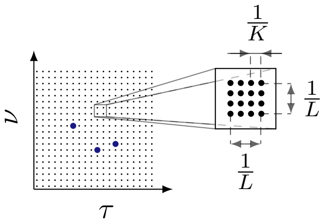
However: Cost of solving the semidefinite programming relaxation is high!

Practical implementation by discretizing the domain

Suppose the (τ_j, ν_j) lie on a **fine grid:**



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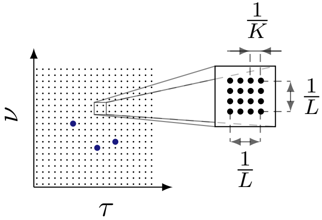

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Structure can be used to accelerate this approach

Stability

Recovery via basis pursuit denoising:

$$\underset{\mathbf{b}}{\text{minimize}} \sum_j |b_j| \text{ subject to } \left\| \mathbf{y} - \sum_j b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x} \right\|_2 \leq \delta$$

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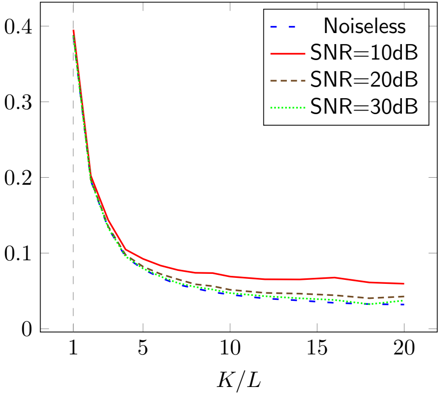
Resolution error:
$L\sqrt{(\hat{\tau}_j - \tau_j)^2 + (\hat{\nu}_j - \nu_j)^2}$



Conclusion

Problem: Estimate the **continuous** time-frequency shifts (τ_j, ν_j) from

$$\mathbf{y} = \sum_{j=1}^{S} b_j \mathcal{F}_{\nu_j} \mathcal{T}_{\tau_j} \mathbf{x}$$

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Thank you!